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Non-classical symmetries and the singular manifold method: a further two examples

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Abstract. This paper discusses two equations with the conditional Painlevé property. The usefulness of the singular manifold method as a tool for determining the non-classical symmetries that reduce the equations to ordinary differential equations with the Painlevé property is confirmed once more. The examples considered in this paper are particularly interesting because they have recently been proposed by other authors as counterexamples of the conjecture made by the authors that the singular manifold method allows us to identify non-classical symmetries. We demonstrate here that the conjecture still holds for these two cases as well. A detailed study of the way of solving this apparent contradiction is offered.

1. Introduction

In 1995 [10] the present authors developed a method for identifying the non-classical symmetries of partial differential equations (PDEs), using the Painlevé analysis as a tool [19] and, more precisely, the singular manifold method (SMM) based on the Painlevé property (PP) [13, 17]. This paper was the continuation of two previous papers [8, 9] by one of us. In it, we studied six different PDEs. Four of them were equations with the PP while the other two considered there were equations with only the conditional PP. The results obtained for these equations can be summarized as the following conjecture: '*The SMM allows one to identify the symmetries that reduce the original equation to an ODE with the Painlevé property*'. Obviously, the combination of this statement with the Ablowitz, Ramani and Segur (ARS) conjecture [1] means that for equations with the PP, the SMM should identify all the non-classical symmetries. Nevertheless, for equations with the conditional PP, the SMM is only able to identify the symmetries for which the associated reduced ordinary differential equations (ODEs) are of Painlevé type.

Recently, Tanriver and Roy Choudhury [16] have applied our method to a family of Cahn–Hilliard equations. According to these authors, their results are apparently in contradiction with ours because (according to them) for these equations the symmetries obtained using the SMM are different from those obtained by the group theoretical non-classical method [14].

If the conclusions of Tanriver and Choudhury [16] were correct, the Cahn–Hilliard equations would be a counterexample that would cast some doubt on the correctness of our conjecture [10].

In the following sections we shall prove that [16] is incomplete and, consequently, that the conclusions of those authors are flawed. When the exercise is done correctly, the results

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show that the Cahn–Hilliard equations are a further two good examples to be added to the list reported in [10].

2. Cahn–Hilliard equation for $m = 1$ and one spatial variable

This equation can be written as [16]

$$u_t + \left(ku_{xx} - \frac{u^2}{2}\right)_{xx} = 0. \tag{2.1}$$

2.1. Non-classical method

The infinitesimal form of the Lie transformation of a PDE with two independent variables x and t can be written as

$$\begin{aligned} x' &= x + \varepsilon\xi(x,t,u) + \mathrm{O}(\varepsilon^2) \\ t' &= t + \varepsilon\tau(x,t,u) + \mathrm{O}(\varepsilon^2) \\ u' &= u + \varepsilon\eta(x,t,u) + \mathrm{O}(\varepsilon^2) \end{aligned} \tag{2.2}$$

such that the associated Lie algebra contains vector fields of the form

$$v = \xi\frac{\partial}{\partial_x} + \tau\frac{\partial}{\partial_t} + \eta\frac{\partial}{\partial_u}. \tag{2.3}$$

The non-classical method [2, 3, 14, 15, 11] requires that the symmetries should obey the invariant surface condition,

$$\xi(x,t,u)u_x + \tau(x,t,u)u_t = \eta(x,t,u) \tag{2.4}$$

associated with the vector field v.

The algorithmic method used to determine the equations to be satisfied by the infinitesimals ξ, η and τ is well known [4, 12, 7]. Nevertheless, as was mentioned several times in [10], the non-classical method requires that the symmetries with $\tau = 0$ should be determined separately from those with $\tau \neq 0$ [5, 6]. Furthermore, there is no restriction in the use of the normalization $\xi = 1$ when $\tau = 0$. In the same way, τ could be normalized to 1 when $\tau \neq 0$ [5, 6].

2.1.1. Symmetries with $\tau = 0$. In this case, we can choose $\xi = 1$ without restriction, which means that the invariant surface condition is $\eta = u_x$.

The equation for η is

$$\begin{aligned} &k\eta_{xxxx} + 4k\eta\eta_{xxxu} + 6k\eta^2\eta_{xxuu} + 4k\eta^3\eta_{xuuu} + k\eta^4\eta_{uuuu} + 6k\eta_x\eta_{xxu} + 6k\eta\eta_u\eta_{xxu} \\ &\quad + 12k\eta^2\eta_u\eta_{xuu} + 12k\eta\eta_x\eta_{xuu} + 6k\eta^2\eta_x\eta_{uuu} + 6k\eta^3\eta_u\eta_{uuu} + 8k\eta\eta_{xu}^2 \\ &\quad + 4k\eta_{xu}\eta_{xx} + 4k\eta^3\eta_{uu}^2 + 3k\eta_{uu}\eta_x^2 + 7k\eta^2\eta_u^2\eta_{uu} + 4k\eta\eta_u^2\eta_{xu} \\ &\quad + 12k\eta^2\eta_{xu}\eta_{uu} + 4k\eta_u\eta_x\eta_{xu} + 10k\eta\eta_x\eta_u\eta_{uu} + 4k\eta\eta_{uu}\eta_{xx} - u\eta^2\eta_{uu} \\ &\quad - 2u\eta\eta_{xu} - u\eta_{xx} - 2\eta^2\eta_u - 3\eta\eta_x + \eta_t = 0. \end{aligned} \tag{2.5}$$

This equation was obtained by using the `symmgrp.max` MACSYMA package [7]. The evident complexity of this equation could be the reason why some authors [16] have neglected these symmetries. This complexity appears for many $\tau = 0$ symmetries [5]. Nevertheless, as will be seen later on, one of the advantages of the SMM is that it provides non-trivial solutions for (2.5).

2.1.2. Symmetries with $\tau \neq 0$. Calculation of these symmetries [7, 16] yields

$$\begin{aligned} \tau &= 4\alpha t + \gamma \\ \xi &= \alpha x + \beta \\ \eta &= -2\alpha u. \end{aligned} \tag{2.6}$$

It is not difficult (see the appendix) to check that the reduced ODEs associated with symmetries (2.6) have the PP only in the following case

$$\alpha = 0 \qquad \beta = 0 \qquad \Longrightarrow \qquad \tau = 1 \qquad \xi = 0 \qquad \eta = 0 \tag{2.7}$$

where γ has been normalized to 1.

2.2. Singular manifold method

Equation (2.1) does not have the PP. However, it is possible to use the SMM to determine particular solutions of (2.1) single-valued on the initial conditions. For such solutions, the equation has the conditional PP. To apply the SMM [17, 19] we should look for solutions of (2.1) in the following form:

$$u = \sum_{j=0}^{\alpha} u_j \phi^{j-\alpha} \tag{2.8}$$

where α and u_0 are respectively the leading index and the leading term and ϕ is the singular manifold that allows us to obtain truncated solutions such as in (2.8). Substitution of (2.8) in (2.1) provides two different expansions that depend on whether the singular manifold is characteristic ($\phi_x = 0$) or not ($\phi_x \neq 0$). We shall explore both cases separately.

2.2.1. Non-characteristic manifold. If $\phi_x \neq 0$, the expansion (2.8) is [16]

$$u' = u - 12k \left(\frac{\phi_x}{\phi}\right)_x \tag{2.9}$$

where u is a solution of (2.1) that could be expressed in terms of the singular manifold as

$$u = 4ks + 3kv^2 \tag{2.10}$$

with v, s and w defined as

$$\begin{aligned} v &= \frac{\phi_{xx}}{\phi_x} \\ s &= v_x - \frac{v^2}{2} \\ w &= \frac{\phi_t}{\phi_x}. \end{aligned} \tag{2.11}$$

It is worth noting that w and s are homographic invariants as opposed to v, which is not invariant under homographic transformations.

The equations of the singular manifold are the equations satisfied by the homographic invariants w and s and are

$$\begin{aligned} w &= 0 \\ s_x &= s_t = 0. \end{aligned} \tag{2.12}$$

The derivatives of (2.10) can be written in terms of the singular manifold as

$$\begin{aligned} u_x &= v(u + 2ks) \\ u_t &= 0 \end{aligned} \tag{2.13}$$

where, according to [10], v^2 has been removed by using (2.10). Substitution of (2.13) in the invariant surface condition (2.4) is

$$v(u + 2ks)\xi = \eta. \tag{2.14}$$

The theory presented in [10] requires that the invariant surface condition should only depend on *homographic invariants*. The infinitesimals must be determined in order to avoid the presence of v in (2.14). The only possibility of eliminating the dependence on v of the invariant surface condition is that $\xi = 0$. This means that the only non-identically zero symmetry is

$$\tau = 1 \qquad \xi = 0 \qquad \eta = 0 \tag{2.15}$$

which is the non-classical symmetry (2.7).

2.2.2. Characteristic manifold. When $\phi_x = 0$, the truncated expansion (2.8) is [18]

$$u' = u - \tfrac{1}{6}(x + x_0)^2 \frac{\phi_t}{\phi}$$

where u is a solution of (2.1) whose expression in terms of the singular manifold is

$$u = \frac{(x + x_0)^2}{12} q(t) \tag{2.16}$$

and where $q(t)$ has been defined as

$$q(t) = \frac{\phi_{tt}}{\phi_t}. \tag{2.17}$$

Notice that for characteristic manifolds [10] the only homographic invariant that we can construct is the Schwartzian derivative with respect to t, defined as

$$h = q_t - \frac{q^2}{2} \tag{2.18}$$

in terms of which, the singular manifold equations are

$$h = 0. \tag{2.19}$$

The derivatives of (2.16) are

$$\begin{aligned} u_x &= \frac{q}{6}(x + x_0) \\ u_t &= \frac{q_t}{12}(x + x_0)^2. \end{aligned} \tag{2.20}$$

(2.16) has to be used to remove q, or $(x + x_0)^2$. The result is

$$\begin{aligned} u_x &= \frac{2u}{(x + x_0)} \\ u_t &= \frac{qu}{2}. \end{aligned} \tag{2.21}$$

Since q is not homographic invariant, we require that τ should be equal to zero in order to avoid its presence in the invariant surface condition (2.4). The infinitesimals are in such a case

$$\begin{aligned} \tau &= 0 \\ \xi &= 1 \\ \eta &= \frac{2u}{(x + x_0)}. \end{aligned} \tag{2.22}$$

It is easy to check that this symmetry satisfies equation (2.5) for the non-classical symmetries with $\tau = 0$.

2.3. Comparison of the non-classical method and SMM

The SMM has allowed us to determine two different symmetries that are, respectively, (2.15) and (2.22). The former is the particular case of the non-classical symmetry (2.7) for which the associated reduction leads to an ODE with PP. The latter is a solution of equation (2.5) for the non-classical symmetries with $\tau = 0$.

These results are in concordance with [10]. As we stated in the last example of this reference '*[The symmetry identified by the SMM]... is the only one in which the associated similarity reduction leads to an ODE of Painlevé type*'.

3. Cahn–Hilliard equation for $m = 2$ and one spatial variable

This equation can be written as [16]

$$u_t + \left(ku_{xx} - \frac{u^3}{3}\right)_{xx} = 0. \tag{3.1}$$

3.1. Non-classical method

In order to properly apply the non-classical method to equation (3.1), we consider two different cases separately.

3.1.1. Symmetries with $\tau = 0$. If $\tau = 0$, we can set $\xi = 1$ with no loss of generality. The resulting equation for η obtained using [7] is

$$\begin{aligned}4k\eta\eta_{xxxu} &+ k\eta_{xxxx} + k\eta^4\eta_{uuuu} + 4k\eta^3\eta_{xuuu} + 6k\eta^2\eta_{xxuu} + 6k\eta_x\eta_{xxu} + 6k\eta^2\eta_x\eta_{uuu} \\ &+12k\eta\eta_x\eta_{xuu} + 6k\eta^3\eta_u\eta_{uuu} + 12k\eta^2\eta_u\eta_{xuu} + 6k\eta\eta_u\eta_{xxu} - u^2\eta_{xx} \\ &-u^2\eta^2\eta_{uu} - 2u^2\eta\eta_{xu} + 4k\eta_{xx}\eta_{xu} + 3k\eta_x^2\eta_{uu} + 4k\eta^3\eta_{uu}^2 + 8k\eta\eta_{xu}^2 \\ &+4k\eta\eta_{xx}\eta_{uu} + 10k\eta\eta_x\eta_u\eta_{uu} + 4k\eta_x\eta_u\eta_{xu} + 12k\eta^2\eta_{xu}\eta_{uu} + 7k\eta^2\eta_u^2\eta_{uu} \\ &+4k\eta\eta_u^2\eta_{xu} - 6u\eta\eta_x - 4u\eta^2\eta_u - 2\eta^3 + \eta_t = 0.\end{aligned} \tag{3.2}$$

3.1.2. Symmetries with $\tau \neq 0$. Solving the system of determining equations obtained using `symmgrp.max` [7] yields [16]

$$\begin{aligned}\tau &= 4\alpha t + \gamma \\ \xi &= \alpha x + \beta \\ \eta &= -\alpha u.\end{aligned} \tag{3.3}$$

It can be shown (see the appendix) that the reduced equations associated with the symmetries with infinitesimal generators (3.3) only have the PP for the special choice of the parameters $\alpha = 0$ and $\beta = 0$. In this case the infinitesimals are simply

$$\tau = 1 \qquad \xi = 0 \qquad \eta = 0 \tag{3.4}$$

where we have set $\gamma = 1$ with no loss of generality.

3.2. The singular manifold method

Equation (3.1) does not have the PP as in the previous equation (2.1). However, it is possible, using the SMM, to search for particular solutions of (3.1) that are single-valued in the initial conditions. We therefore seek solutions of the form [17, 16]

$$u' = \sum_{j=0}^{\alpha} u_j \phi^{j-\alpha}. \tag{3.5}$$

The leading index is an integer only when the singular manifold ϕ is non-characteristic ($\phi_x \neq 0$), in which case expansion (3.5) takes the form [16] of

$$u' = u + \sqrt{6k}\left(\frac{\phi_x}{\phi}\right) \tag{3.6}$$

where u is a solution of (3.1) that is expressed in terms of the singular manifold ϕ as

$$u = -\frac{\sqrt{6k}}{2} v. \tag{3.7}$$

Furthermore, the singular manifold equations that relate w and s are

$$\begin{aligned} w &= 0 \\ s &= 0. \end{aligned} \tag{3.8}$$

The next step is to compute the derivatives of (3.7) in terms of the singular manifold. The result is

$$\begin{aligned} u_x &= -\frac{1}{\sqrt{6k}} u^2 \\ u_t &= 0 \end{aligned} \tag{3.9}$$

where [10] v^2 has been eliminated using (3.8). Substitution of (3.9) in the invariant surface condition gives

$$-\frac{1}{\sqrt{6k}} u^2 \xi = \eta. \tag{3.10}$$

According to equation (3.10) we must consider two cases.

- $\xi = 0$. In this case $\eta = 0$ and the only nontrivial symmetry we obtain is

$$\tau = 1 \qquad \xi = 0 \qquad \eta = 0 \tag{3.11}$$

which corresponds to the non-classical symmetry (3.4).

- $\xi = 1$. In this case equation (3.10) is the invariant surface condition associated with a symmetry with $\tau = 0$. The infinitesimal generators then take the form

$$\tau = 0 \qquad \xi = 1 \qquad \eta = -\frac{1}{\sqrt{6k}} u^2. \tag{3.12}$$

It is trivial to check that (3.12) satisfies equation (3.2) for the non-classical symmetries with $\tau = 0$.

4. Conclusions

• Two equations with the conditional PP have been considered. Here we show that both of them satisfy the conjecture established in [10] to the effect that the SMM allows us to identify all the non-classical symmetries that reduce the equation to an ODE with the PP.

• Our results do not agree with those found recently by Tanriver and Chowdhury [16]. This is because, although these authors have tried to follow the method discussed in [10], they fail to consider some of the aspects that were clearly stated in this reference. Such omissions lead them to wrong conclusions and can be listed as follows.

(1) They do not take into account in both examples that computing the non-classical symmetries of an equation requires the consideration of two different cases separately, namely $\tau = 0$ and $\tau \neq 0$. Only symmetries with $\tau \neq 0$ were evaluated in [16]. However, as we have shown in this paper, some of the symmetries of the solutions found using the singular manifold method are symmetries with $\tau = 0$.

(2) On applying the singular manifold method, they do not consider the case in which the singular manifold is characteristic ($\phi_x = 0$). Solutions evaluated on the basis of characteristic manifolds turn out to be relevant for equation (2.1), just as was the case of some of the examples analysed in [10].

(3) The authors of [16] should submit [10] to careful scrutiny. In the introduction to [10] the following explicit statement was made.

> *'We show how for PDE with Painlevé property, these symmetries are precisely those obtained through the non-classical method... Finally, for equations with the conditional Painlevé property, the SMM allows one to identify the symmetries that reduce the original equation to an ODE with the Painlevé property.'*

According to the last sentence, and since equations (2.1) and (3.1) have the conditional PP, the second part of the sentence applies for both of them. As has been shown for these examples (as well as was shown for example 6 in [10]), the non-classical symmetries that cannot be recovered through the SMM are precisely those that reduce the equation to ODEs that are not of Painlevé type.

• We believe that we have shown here that both equations, (2.1) and (3.1), have not been interpreted correctly in [16]. Careful analysis shows that the procedure developed in [10] merely provides two more examples that confirm the relationship between the non-classical method and the SMM.

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Appendix

A.1. Non-classical reductions for symmetries (2.6)

From the non-classical infinitesimal generators (2.6) we obtain two different reductions.

• If $\alpha \neq 0$

$$u = \frac{\alpha^2}{(\alpha x + \beta)^2} F(z)$$

$$z = \frac{(\alpha x + \beta)^4}{a^3(4\alpha t + \gamma)}$$

where $F(z)$ must satisfy the ODE:

$$4k[32z^4 F_{zzzz} + 80z^3 F_{zzz} + 30z^2 F_{zz} - 15z F_z + 15F]$$
$$-8z^2 F F_{zz} - 8z^2 F_z^2 + 10z F F_z - 2z^2 F_z - 5F^2 = 0$$

which is not of Painlevé type.

- If $\alpha = 0$

$$u = F(z)$$
$$z = \gamma x - \beta t$$

where $F(z)$ is a solution of

$$-\beta F_z + k\gamma^4 F_{zzzz} - \gamma^2 (F F_{zz} + F_z^2) = 0.$$

It can be easily shown that this equation is not of Painlevé type unless $\beta = 0$.

A.2. Non-classical reductions for symmetries (3.3)

Similarly, we obtain two different reductions from the symmetry (3.3).

- If $\alpha \neq 0$

$$u = \frac{\alpha}{(\alpha x + \beta)} F(z)$$
$$z = \frac{(\alpha x + \beta)^4}{a^3(4\alpha t + \gamma)}$$

where $F(z)$ must satisfy

$$4k[64z^4 F_{zzzz} + 224z^3 F_{zzz} + 108z^2 F_{zz} - 6z F_z + 6F]$$
$$-16z^2 F^2 F_{zz} + 12z F^2 F_2 - z^2 F_z - 32z^2 F F_z - 4F^3 = 0$$

which is not of Painlevé type.

- If $\alpha = 0$

$$u = F(z)$$
$$z = \gamma x - \beta t$$

where $F(z)$ is a solution of

$$-\beta F_z + k\gamma^4 F_{zzzz} - \gamma^2 (F^2 F_{zz} + 2F F_z^2) = 0.$$

This equation is again of Painlevé type only when $\beta = 0$.

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